

COMMENTARY

On Motivating Higher-Order Logic

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PROFESSOR HIGGINBOTHAM ASKS whether higher-order logic may be adequately motivated by the properties of natural languages, and he concludes that it cannot. I have no quarrel with this conclusion, but that is because I think he is looking in the wrong place for a suitable 'motivation', and that that in turn is because he has a mistaken conception of what is distinctive about higher-order logic. To make this point as straightforwardly as possible, I shall begin by concentrating just on second-order logic, and in fact on just that fragment of it that Higginbotham himself gives most attention to. Later, I shall briefly sketch the wider picture.

1. What is second-order logic?

It is natural to begin by contrasting the standard vocabulary of first-order and second-order logic, as they are usually conceived. I would put it in this way. In the vocabulary of first-order logic there are (i) schematic letters which we think of as taking the place of names or other subject-expressions ('name-letters'), (ii) schematic letters to take the place of (first-level) predicates ('predicate-letters'), which form an atomic formula when followed by suitably many name-letters, and (iii) variables ('pro-names'?) which take the place of the schematic name-letters, but must then be bound by quantifiers to obtain a closed formula. There may also be schematic function-letters; I shall ignore

these. There will also be some truth-functors, but these will play no part in the discussion. What I take to be the standard vocabulary of second-order logic makes two additions to this: first, variables ('pro-predicates'?) which take the place of the schematic predicate-letters, and must be bound by quantifiers in order to obtain a closed formula; and second, a new type of schematic letter, which we think of as taking the place of *second-level* predicates.

It is to be noted here that, if we follow Frege's way of doing things, the addition of a new letter for second-level predicates does not alter the syntax of the first-level ones. For example, the formula $\forall xFx$ is now seen to represent a particular second-level predicate, namely the universal quantifier, applied to the first-level predicate here represented by F . And when we introduce a new schematic letter, say M , to represent an arbitrary second-level predicate, it will appear in contexts such as $MxFx$. Thus the letter F still occurs in immediate concatenation with a variable (or variables) of lowest level. In the perhaps more familiar notation of Church's λ -calculus, one writes rather $M(\lambda x . Fx)$, and this variant notation does in practice have some technical advantages, e.g. in stating the substitution rules. But it is merely a variant notation, and if it is adopted then it will also be natural to write the first formula above as $\forall(\lambda x . Fx)$. The extended vocabulary, then, allows first-level predicate-letters to occur 'in argument position' only to the extent that they already so occur in the familiar formula $\forall xFx$.

In any case, we can for present purposes set aside any further consideration of how to arrange for these new schematic letters, for it is clear that the second-order logic that Higginbotham mainly considers does not contain them. This is because he is concerned with relations between logic and natural language, and it seems very reasonable to say as he does (§1) that a natural language does not contain schematic letters of *any* kind. (The point is perhaps not evident on his account of what we may reasonably regard as a 'natural' language. For example, Aristotle falls into using such letters, without any special explanation, when expounding his syllogistic. Why should one say that the Greek he is then employing is no longer his 'first language', nor even a possible 'first language' for anyone?) We can similarly set aside the provision of predicate-variables for all predicates except the simplest, namely the one-place predicates, for again Higginbotham himself allows that quantification over predicates of two or more places is subject to special restraints of its own, which he has explored elsewhere (in particular his 1990). The result is that the second-order logic we have to consider

differs from ordinary first-order logic *only* in that it contains variables for one-place predicates, and of course quantifiers to bind them. How, then, does it differ from a theory of classes or sets which is intended to allow us to quantify both over some objects which we will call 'individuals', and over classes of them? (I shall speak mainly of classes rather than sets, to allow for the possibility of taking the 'individuals' to include *all* sets, for in that case our classes of them would have to be what are called 'proper classes', not sets.) To make the comparison closer, we may suppose that the class theory is also a 'two-sorted' theory, with one sort of letter and associated variable for individuals, and another sort for classes of these individuals. How, then, are the two theories to be distinguished? Higginbotham spends some time discussing this question, which is raised by Azzouni (1994), but it seems to me that he gives the wrong answer to it.

I think we can agree that the mere fact that in one theory one writes Fa , and in the other $a \in F$ is not by itself of any significance. It could be no more than a variation in notation, much as Polish notation for the truth functions is a variation on that standardly used outside Poland. A point which may seem to be of more significance is that the two logics may differ on what they count as a well-formed formula. For in the second-order predicate logic such concatenations of symbols as ab and FG will not be counted as formulae, whereas the class theory may well accept the analogous concatenations $b \in a$ and $G \in F$ as perfectly well-formed. But one may reply that, while our horizons are restricted as at present (I shall widen them in §3), the difference does not in fact seem to be of much importance. For though our class theory may indeed allow $b \in a$, $G \in F$ and the like as well-formed, still it must condemn all such formulae as false, and false simply because of their syntax. But in that case what would be lost if we did not count them as formulae at all? (It may be noted that when Russell first put forward a theory of types, in his *Principles of Mathematics*, he was certainly a realist about the existence of classes. Yet his proposed theory of classes contained just those restrictions on what is to count as a well-formed formula that we now think of as characteristic of higher-order logic.) I conclude that the difference between the two systems, if there is one, need not show up in their respective syntax, either in their rules of formation or—as I now add—in their rules of proof. So it must, apparently, lie in their semantics.

On this point Higginbotham agrees. For he argues that the substitutional interpretation of the predicate-variables is too weak to be of

interest, and that second-order logic must provide some range of *values* for these variables. Moreover, he argues that these values must be taken to be 'unsaturated' entities, as Frege's concepts are supposed to be, whereas the class theory will of course say that the values of the class-variables are to be classes, and from Frege's viewpoint classes are objects, and hence 'saturated'. But in my view this talk of different kinds of entities which one may take as 'values' is unhelpful.

Second-order logic must of course provide *truth-conditions* for its second-level quantifications, but it seems to me (as it seemed to Boolos 1975) that we need not see these as providing any entities—saturated or unsaturated—for the variables to range over. On the contrary, even in first-order logic we are familiar with the idea of considering 'all interpretations' of our schematic letters, both name-letters and predicate-letters, for the key concept of validity is defined as truth (or truth-preservation) in all interpretations. But the truth-conditions for the quantifiers can be given by using just the same idea. Thus a first-level quantification $\forall x(\text{---}x\text{---})$ is counted as true (in a given interpretation) if and only if its singular instance $(\text{---}a\text{---})$ is true for all interpretations of a , provided that a is a new name-letter not already occurring in $(\text{---}x\text{---})$, and retaining the given domain and the given interpretations of all other symbols. In an exactly similar way, a second level quantification $\forall F(\text{---}F\text{---})$ is counted as true if and only if its singular instance $(\text{---}G\text{---})$ is true for all interpretations of G , under the same proviso. A name-letter is 'interpreted' as denoting some object in the given domain of objects, and a (one-place) predicate-letter is 'interpreted' as true of some objects in that domain (or none), and false of the others. So to speak of all ways of interpreting a predicate-letter is just to speak of all ways in which it may be true of some objects in the domain and false of others (and in each case we do not restrict attention to interpretations that can be specified within the language under consideration). Thus in second-order logic, as in first-order logic, the only 'interpretations' considered are extensional, i.e. they merely assign a truth-value to a sentence-letter, an object of the domain to be the denotation of a name-letter, and some objects of the domain to be those the predicate is true of. This is all we need for the characterisation of validity in these logics. In other kinds of logic, such as modal or epistemic logic, an interpretation may have to be more intensional, e.g. it may have to assign a condition under which the name is to denote an object or the predicate is to be true of one. But it will still be the case that interpreting a predicate-letter does *not* involve assigning to it any entity that it refers to.

The point I wish to make is this. Where the letter F is interpreted as true of just such-and-such objects, one *can* of course say that this provides a singular entity, namely the class of all those objects, to be its 'value'. Or, if one believes in plural entities—as Higginbotham apparently does, at least if we are to take seriously his endorsement of Russell's 'classes as many'—one can instead say that it provides a plural entity, namely those objects, as the 'value' of that letter. Or again, if one believes in Fregean concepts, one can say that it provides such a concept as the 'value', namely the ('unsaturated') concept that maps each of those objects into the True and everything else into the False. If we grant the existence of entities of these three kinds, there seems nothing to choose between the various accounts. (And in place of classes we could, of course, speak of sets, provided that our domain of objects does constitute a set.) But the important point is that even if one does not believe in the existence of sets or classes or plural entities or Fregean concepts—and many do not—one can *still* speak of interpreting F as true of just such-and-such objects in the domain. There is no need to invoke any such controversial entities as 'values', and certainly there is no need to treat a predicate-letter as *referring* to such a value, as Frege apparently did. For predicates do not refer; rather, they are true of some things and false of others. I believe, then, that it is a complete mistake to say that what is distinctive of second-order logic is that it invokes a special kind of entity to be the value of a predicate-variable. For the conceptual resources needed to explain these variables are just the same as are needed to explain first-order logic, and they need not be seen as requiring any such special entities. Of course I do not deny that it is highly convenient to use the familiar terminology of classes or sets, and most textbooks do so. But it is not essential.

So let us come back to the question: what *does* make the difference between a logic of first order and of second order? The important point is surely that the notion of *validity* works out differently in the two cases. Of course, the formal definition is the same in both: a formula (or sequent) is valid if and only if it is true (or truth-preserving) under all permitted interpretations. In each case, we permit only certain interpretations of the truth-functors, namely the intended ones. In a first-order logic we also stipulate a special interpretation for the first-level quantifiers, namely that they are to be taken as generalising over *all* objects in the domain (whether or not we have names for those objects). But in a second-order logic we stipulate in addition a special interpretation for the second-level quantifiers, namely that they are to be taken as

generalising over all ways in which a predicate may be true of some objects in the domain and false of the others (and here again we mean *all* such ways, not only those which the language in question happens to have words for). It is this extra clause in the definition of what counts as a permitted interpretation, and hence in the definition of validity, that creates the important differences between the two logics. For example, it is responsible for the fact that first-order logic may be provided with a complete set of rules and axioms, whereas second-order logic has no complete proof procedure; the fact that first-order logic is compact, whereas second-order logic is not; the fact that many important notions not definable in first-order logic are definable in second-order logic; and so on.

I give just one simple illustration of this last point. It is well-known that identity is not definable in any first-order theory, i.e. if the theory has 'normal' models (in which '=' is interpreted as identity) then it also has 'non-normal' models (in which '=' is differently interpreted). To see this, one need only observe that, given any normal model for the theory, we can create from it a non-normal one in this way: simply split each item in the domain into two 'twins', with each twin in the new model satisfying exactly the same predicates as its parent in the old one, but with '=' interpreted as holding not only between an object and itself but also between an object and its twin. The result must be a non-normal model for the same theory. And this holds whatever special rules or axioms for '=' are laid down in the theory, provided that the only quantifiers available in the theory are the first-level quantifiers. But if second-level quantifiers are present then of course identity can be defined by $a = b \leftrightarrow \forall F(Fa \leftrightarrow Fb)$. This must have the desired effect, since one permitted interpretation of F is to take it as true of the object denoted by a and of nothing else. A similar point holds for many other notions besides identity; I shall exploit one in the next section.

The important point about a second-order logic, then, is that it has an extra kind of quantification, on which a special interpretation is imposed. In consequence it has much greater expressive power, and at the same time it is no longer axiomatizable or compact, as first-order logic is. Of course we can achieve exactly the same effect in the idiom of the theory of classes. If, as envisaged, the class theory is presented as a two-sorted theory, with one sort of variable for individuals and another for classes of them, and if further we stipulate the appropriate interpretations in each case, then there really is no significant difference between them, for whatever can be done in the one can equally be done in the other. They may perfectly well be regarded merely as notational

variants of one another, and I would classify each as a second-order theory. Somebody may insist that the intended interpretation of the class-quantifiers is as generalising over classes, whereas we have seen that the interpretation of the predicate-quantifiers does not have to be given in these terms, but this will be a significant distinction only for one who holds that the existence of these classes (which may include classes that we cannot specify) is seriously in doubt. For those who are not so squeamish about their ontology, the complete equivalence of the two logics—in matters of proof, validity, expressive power, and so on—will make it appear that *at this stage* there is nothing to choose between them. The choice will take on a genuine significance when we progress to higher orders, as I shall do (very briefly) in my §3. But before I come to this I need to say something about ‘motivations’ for second-order logic, as described so far.

2. *Motivations for second-order logic*

Logic concerns reasoning. This has no simple and direct connection with the linguistic structures of natural languages. I offer just two brief examples. First, Frege’s greatest claim to fame (in my view) is his discovery of what we now call ‘modern logic’, in his *Begriffsschrift* of 1879. And the central and most important innovation in the *Begriffsschrift* is its treatment of quantification. Why did it take so long for the modern view of quantification to be discovered? For there are 22 centuries between Aristotle and Frege. The answer is surely that in natural languages the means for expressing quantification are strange and convoluted. Thus in English to make clear the scopes of our quantifiers we rely on differences between ‘every’, ‘any’, and ‘all’, on the distinction between the active and the passive voice, and other such *ad hoc* devices. From the modern viewpoint, this is an extremely clumsy way of doing the job, and if we are interested in reasoning we shall clearly see the way it is done in modern logic as both superior and very different. It took a genius to discover this modern way just because the structures of natural languages do not in any way suggest it. Second, a related but much more general point. It is agreed that on a reasonable conception of what is to count as a natural language, such languages do not include the use of letters to express generality (either as schematic letters or as bound variables). But they are an indispensable aid to the expression of long and complex trains of reasoning, and so have in fact

been used for centuries by people (especially mathematicians) engaged in such reasoning. But it would be absurd to ask whether this use of letters is adequately 'motivated' by features of natural languages, for the motivation is rather the need to transcend natural languages, in order to think and reason more clearly.

If, then, we ask for a 'motivation' for second-order logic, and expect this motivation to come from more ordinary procedures, the right place to look is to the way in which people do actually reason. Here the first thing to say is that ordinary first-order logic covers an amazing amount of our ordinary reasoning, and the need for a second order only emerges at a rather sophisticated stage. But the second thing to say is that it does, eventually, emerge. The clearest examples come from mathematics, and I briefly cite just two, which are reasonably well known. Dedekind in effect discovered a set of postulates for the elementary arithmetic of natural numbers (the postulates generally known today as 'Peano's postulates'), and he gave a proof that they are 'categorical' (i.e. that all models of them have the same structure, in this case the structure of the natural numbers). Cantor similarly put forward a set of postulates for the real numbers, and gave a proof of their categoricity. But each of these proofs relies upon a second-order understanding of the postulates in question, for it is well known that no set of first-order postulates which has an infinite normal model can be categorical. I, for one, would wish to accept these proofs as correct, so this commits me to accepting second-order logic.

These examples are somewhat *recherché*, so it may be useful if I extract from the first of them something rather less technical. Consider the argument which has infinitely many premises, as follows:

a is not a parent of *b*
a is not a parent of a parent of *b*
a is not a parent of a parent of a parent of *b*
 &c

From all these infinitely many premises taken together there follows the conclusion

a is not an ancestor of *b*

Of course, the argument can be recognised to be valid only by one who understands the relationship between being a parent of and being an ancestor of. Now since first-order logic is compact, it cannot recognise this argument as valid. For in a compact logic whatever follows from an infinite set of premises must also follow from some finite subset of

them, and that is evidently not the case here. One must infer that the required relationship between being a parent and being an ancestor cannot be formulated in first-order terms.¹ But it can be formulated if the second-level quantifiers are available: that is just what Frege showed when he showed how to define the ancestral of any relation. In second-order logic we need only add this definition to the premises, and then there is no difficulty in showing that the argument is valid. This seems to me good ground for saying that even quite ordinary reasoning can involve resources which are available in a second-order logic, but not in a first-order logic. But I imagine that plausible examples would have to involve infinity in some way, as this one does, for that is where the limitations of first-order logic first become clear.

There are those who do not accept second-order logic, or anyway do not accept it *as logic*. In some cases, notably Quine's, the objection is specifically to the predicate logic version, and the main ground of the objection seems to be that this is not English. Of course I agree that it is not English, but it seems to me none the worse for that. In any case, this objection does not apply to the version which uses the idiom of classes, but which I am prepared to treat as just the same theory in a different notation. A more common line of objection, found especially amongst those concerned with mathematics, is the old *horror infiniti* in a new guise. In particular, it is said that we do not really understand the notion of *all* subsets of an infinite set, or (equivalently) the notion of *all* ways of interpreting a predicate-letter on an infinite domain. (I imagine that the fact that Cantor's continuum problem is still unsolved has a lot to do with this proclaimed lack of understanding.) Hence we do not understand the second-level quantifiers. All I can say to this is that I feel no such lack of understanding myself. It is similarly all one can say to committed intuitionists, who claim that they do not understand the classical concept of negation. Finally, there are those, e.g. Kneale, who say that no matter how good a theory second-order logic may be, still it cannot be counted as *logic* because a genuine logic must have a complete proof procedure. I have some measure of sympathy

¹ It may seem that one can at least give a first-order definition of 'parent' in terms of 'ancestor'. Actually this is not too simple, for a person has two parents, one of whom may also be a grandparent. But let us change the example a little, writing 'mother' for 'parent' and 'matrilineal ancestor' for 'ancestor'. Then one can certainly give a first-order definition of 'mother' in terms of 'matrilineal ancestor'. But adding this definition to the premises still cannot give us an argument that is first-order valid, just because first-order logic is compact.

with this point, for certainly we should all prefer it if a complete proof procedure were available. But one cannot regard it as a cogent argument. For the bounds of logic are indeterminate, and one can equally imagine someone claiming that all genuine logics must have an effective decision procedure. This, of course, would rule out even first-order logic, as we now understand it, but it is a criterion that would be satisfied by all logics before Frege's.

I therefore recommend the acceptance of second-order logic, whatever may or may not be suggested by the idioms of natural languages.

3. *Logics of higher orders*

The second-order predicate logic considered so far is a truncated theory, for it omits quantification over predicates of two or more places, and it lacks schematic letters for second-level predicates. That is why it exactly matches the second-order theory of classes or sets. But let us now restore the omissions. It is clear that with quantification over many-place predicates now available, the predicate logic at once scores over the class logic, for in a theory of classes relations can only be handled in a somewhat roundabout way. (Standardly one identifies the relation R with the class of all unordered pairs $\{\{x\}, \{x, y\}\}$ such that Rxy . But this is a *second*-level class, and so not available in the class theory so far considered.) But a more significant difference comes to light when we reflect upon the restored schematic letters for second-level predicates. For it at once becomes obvious that the step by which we moved from a theory of first order to one of second order, namely by subjecting the first-level predicate-letters to quantification, can now be repeated. So we move up to a third-order theory, by once more introducing quantifiers to bind our second-level predicate-letters, at the same time adding a new kind of schematic letter to represent any arbitrary third-level predicate. Clearly the ascent can continue, and there is no natural stopping place, so we are led on to the full hierarchy of types, with quantifiers of every (finite) level. Now of course something very similar can be done in the class theory, by adding new styles of variables for classes of second level, then of third level and so on, until again an infinite hierarchy is reached. But there are two very significant differences between the two hierarchies.

First, the hierarchy of types will be a *strict* hierarchy, in the sense that items of one level can take as arguments only items of the *next*

lower level, and all other ways of putting symbols together will be deemed meaningless. This restriction has a good motivation in the origins of type theory, namely as developing out of the familiar first-order predicate logic. For names and first-level predicates are evidently expressions of different types, and second-level predicates (for example, the familiar quantifiers) are different again, and so on up. Thus, just as one cannot form a sentence by putting together two names, or two first-level predicates, so also one cannot form a sentence by putting together a quantifier and a name, and so on. To put things more precisely, and remembering that we have relations to consider too (which will now include mixed-level relations), we may set out the position in this way. Each different style of letter and variable will be regarded as carrying a type-index, and the type-indices are defined thus:

- (i) 0 is a type-index (and is the index for names);
- (ii) If α, β, \dots are type-indices, so is (α, β, \dots) .

To illustrate briefly, the first-level types are $(0), (0,0), (0, 0, 0), \dots$; the second-level types include $((0)), ((0, 0)), ((0), (0)), \dots$; and mixed-level types include $((0),0), (((0), 0), (0), 0), \dots$. The ruling on significant combinations is this: an expression of type (α, β, \dots) can take as arguments only a series of expressions of types α, β, \dots respectively. As I have indicated, this ruling is apparently imposed just by very elementary requirements on how to put words or symbols together to form a sentence.

By contrast, grammar imposes no such requirements on the formulae of a theory of classes or sets. For the predicate in such a formula is always \in , i.e. 'is a member of', and all classes of whatever level are thought of as referred to by name-like expressions, and there is no grammatical bar to combining *any* two such expressions with \in . Consequently, while one *can* of course propose a hierarchy of classes which is strict in the same sense as before, and thus corresponds exactly with the monadic part of the type hierarchy, that is not what is usually done, for it seems to leave a lot out. If classes or sets are thought of as hierarchically arranged at all, then nowadays one will take the hierarchy as a *cumulative* one. So one begins, as we have done, with one style of variable for individuals, and another for classes of individuals. But the variables introduced at the next stage will range over all classes that have as members *either* individuals *or* first-level classes *or both*. And generally the variables of level n will range over all the classes that can

be formed from all items of *all* levels less than n . This clearly gives a hierarchy of a very different structure to that of the type hierarchy.

A further, and consequential, difference is this. There is no conceptual bar to extending the cumulative hierarchy of classes into the transfinite, i.e. by adding yet further classes of level ω , which can have as members items from *all* finite levels, and then classes of level $\omega + 1$ which can also have classes of level ω as members, and so on up without limit. But, at least at first sight, there would seem to be no sense in trying to extend the strict hierarchy of types in a similar way. For if each type of expression can take only the next lower type of expression as argument, there is clearly no role for an expression of level ω , and in any case one is quite at a loss to say what an 'infinite type' might be.

We began with a predicate logic, and a corresponding class theory, of second order. At that stage, the theories appeared so like one another, in all important respects, that there was scarcely any question of choosing between them. But we have now seen that when each is extended to yet higher orders, in what seems to be the natural way, they do diverge very considerably. Yet each of them, it seems to me, deserves to be called a higher-order logic. At any rate, they share these features: each is a many-sorted theory, with many different kinds of variable; in each case the intended range of each different kind of variable is explained differently, and hence the intended interpretation of the quantifiers that bind these variables; so what counts as a 'permitted interpretation' is at each stage restricted by the stipulated interpretation for these quantifiers. It follows that the key notion of validity, i.e. truth in all permitted interpretations, is multiply restricted in each case. This, as we saw, was the source of the important differences between an ordinary logic of first order and of second order; it is now reiterated infinitely many times.

There are, no doubt, yet other higher-order theories. In particular, there are *predicative* versions both of our type theory and of our class theory. But I cannot consider them here. Instead, I end with some very brief considerations of what motivation one might have for adopting either of the two theories here outlined.

4. *Motivations again*

One important distinction between the two theories lies in what they do or do not allow one to say. From this point of view, the restrictions imposed by type theory are certainly unwelcome, and I believe that they

should rule it out of consideration. No matter how natural these restrictions may appear to be, when one looks at the motivations for the theory, still their effect is disastrous. A well-known example is that if, like Russell, one cannot accept Frege's claim that numbers are individuals of the lowest level, then type theory provides no level on which to locate them. But the complaint is not restricted to numbers, or even to mathematical entities more generally; it continues to surface with all kinds of other examples, in particular with many concepts employed by logicians. For example, relations of many different types may be transitive, so the predicate 'is transitive' cannot be assigned to any type. I have explored these problems in my (1980), and I shall not further rehearse them here. But I do in that article suggest a way of *extending* type theory, by adding what I call 'type-neutral' predicates, which I think goes a fair way towards resolving this problem. The addition is in some ways similar to the addition to a class theory of classes at infinite levels, but I would not wish the analogy to be pressed at all strictly. If type theory is to survive as a serious claimant on our attention, then I believe it must first be extended in some such way as I have proposed. But there is still room for serious doubt over whether, even when so extended, it is adequate to represent the ordinary reasoning of logicians and mathematicians.

By contrast, the class theory that I have outlined is now more or less orthodox amongst them, though retitled 'set theory', and with two variations from what I have said so far. The first is of no philosophical importance: for technical simplicity mathematicians prefer to work with the theory of *pure* sets, in which there are no individuals and every item is a set, built up ultimately from the null set. The second is more relevant to our present concerns. The sets are regarded as being all of the *same* logical type, and consequently the theory is often presented as a first-order theory, with just the one kind of quantifier, which ranges over all sets. The sets are still arranged in a cumulative hierarchy (this is ensured by the axiom of foundation), stretching into the transfinite, as I have briefly described. To speak very broadly, this is what protects the theory from the well-known 'paradoxes' (i.e. contradictions) that our naive thinking about sets—or, indeed, about other abstract objects—so often falls into. And the example shows very nicely that Russell was wrong in supposing that the *only* way to avoid these contradictions is to pay careful attention to distinctions of logical type. Moreover, the previous objections to type theory now automatically fall away; for example, since sets are all of the same logical type, there is now no

difficulty in defining a predicate 'is transitive' which is true of *all* sets that are transitive relations, whatever their level in the cumulative hierarchy may be. Should we then conclude that there is after all no need for theories of higher orders, not even of second order?

Far from it. My previous considerations remain in force, and can be further illustrated from set theory itself, for when that theory is presented as a merely first-order theory it cannot do justice to our intuitive conception of a set. Like all first-order theories, it must permit 'unintended' models, which we can recognise as a distortion of that original conception. Indeed, the extensive and fruitful work on 'inner models' of set theory is a copious illustration of the point. So set theory itself seems to me to cry out for a second-order formulation, and it is easy to see how this is to be done. For example, in a first-order formulation the familiar axiom of subsets must be laid down as a schema generating infinitely many axioms: for any well-formed formula $A(z)$ containing free occurrences of z but not of y , there is an axiom $\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge A(z))$. In effect, this generalises only over those properties of members z of x which can be expressed in the standard vocabulary of set theory. But we surely wish to generalise over *all* such properties, whether or not they can be so expressed, and the way to do this is obvious: introduce a second-level quantifier, and write $\forall F \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge Fz)$. When the axiom of subsets is superseded by the axiom of replacement, one can of course give a second-level formulation of that axiom too.

Unfortunately one cannot claim that the second-order formulation of set theory results in a categorical theory, for it does nothing to resolve this much-debated question: just how far does the cumulative hierarchy of sets extend? Or, in other words, how far does the series of infinite ordinals go? But it does introduce a definiteness hitherto lacking. I give just one well-known example, Cantor's continuum hypothesis, which states that there is no infinite cardinal number between the number of the natural numbers and the number of the real numbers. As we know, given just the usual first-order formulation of set theory, all one can say about this hypothesis is that it is true in some models of the theory and false in others. But with a second-order formulation one can say more strongly: either it is true in *all* models of the theory, or it is false in *all* models. But, of course, we still do not know *which*. I would say that the fact that people are still trying to resolve this problem shows that it is in fact the second-order formulation of set theory that they are really working with.

So far as I am aware, the second-order formulation of set theory has all the advantages of the theories of yet higher orders previously considered. The addition of 'proper classes' to a standard set theory goes *some* way towards a full second-order formulation, but—as it is usually presented—not all the way. So I conclude that there is a real need for second-level quantification, but whether there is also a need for quantification of higher levels is still a moot question, and further exploration is needed.

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